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ON THE MOTION OF TWO SPHERES IN A PERFECT FLUID

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Motion of two spheres in a perfect incompressible fluid is considered. Kinetic energy and hydrodynamic forces are computed for the case when the distance between the spheres is small, in particular when the spheres touch each other. Singularities arising in the velocity field on contact of the spheres are determined.

Hicks [1] obtained the kinetic energy of the fluid for the spheres moving along the line-of-centers (the line joining the sphere centers). The kinetic energy for the case when the spheres move in the direction perpendicular to the line-of-centers and the distance separating them is much larger than their radii, is known from [2].

1. Velocity potential. Two spheres move in a perfect incompressible fluid which is at rest at infinity. The fluid motion is assumed potential. Since the problem is linear, the case when the velocities of the spheres are coplanar, is sufficient to obtain the velocity potential.

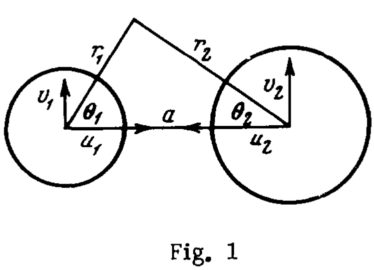

Fig. 1

We choose the spherical system of coordinates r_i, θ_i, φ_i with the origin at the center of the ith sphere ($i = 1, 2$) and the positive directions of their polar axes oriented towards the neighboring sphere (Fig. 1). Azimuthal angle φ_i is measured from the direction perpendicular to the velocities of the spheres, and the positive direction of the polar axis of the ith coordinate system is taken as positive direction of the projection u_i of the velocity on the line-of-centers. Positive directions of the projections v_1 and v_2 of the velocities of the spheres on a line perpendicular to the line-of-centers, are chosen so as to coincide.

Fluid velocity potential Φ should satisfy the Laplace's equation in the region situated outside the spheres, as well as the following boundary conditions:

$$\Delta\Phi = 0, \ \partial\Phi/\partial r_i|_{R_i} = u_i \cos\theta_i + v_i \sin\theta_i \sin\varphi_i$$

$$\Phi \to 0 \quad \text{when} \quad r_i \to \infty$$

where R_i is the sphere radius. Method of images can be used [1-4] to solve the problem. The potential is obtained by the method of consecutive approximations and is represented by the sum of series in terms of functions Φ^i_n

$$\Phi = (\Phi^1_0 + \Phi^1_1 + \Phi^1_2 + \ldots) + (\Phi^2_0 + \Phi^2_1 + \Phi^2_2 + \ldots) \tag{1.1}$$

These functions are harmonic outside the i th sphere and satisfy the following conditions on the i th sphere:

$$\partial\Phi^i_0 / \partial r_i = u_i \cos\theta_i + v_i \sin\theta_i \sin\varphi_i \tag{1.2}$$

$$\partial\Phi^i_n / \partial r_i = -\partial\Phi^k_{n-1} / \partial r_i \quad (n = 1, 2, \ldots) \tag{1.3}$$

$$\Phi^i_n \to 0 \ \text{when} \ r_i \to \infty \quad (n = 0, 1, \ldots)$$

Here and henceforth $k = 1, 2$ and $k \neq i$.

We begin by considering the motion of spheres along the line-of-centers ($v_1 = v_2 = 0$). As we know [1-4], in this case the functions Φ^i_n will represent the potentials of the dipoles situated within the spheres on that line, and we can seek them in the form

$$\Phi^i_n = \alpha^i_n (r_i \cos\theta_i - a_{in}) (r_i^2 - 2r_i a_{in} \cos\theta_i + a_{in}^2)^{-3/2} \tag{1.4}$$

Inserting (1.4) into (1.3) we obtain equations defining the unknown coordinates and the dipole strengths

$$a_{in} (a - a_{kn-1}) = R_i^2, \quad a_{i0} = 0 \tag{1.5}$$

$$\alpha^i_n = \alpha^k_{n-1} (a_{in} / R_i)^3, \quad 2\alpha^i_0 = -u_i R_i^3 \tag{1.6}$$

Here a denotes the distance between the centers of the spheres. These recurrent relations are solved most simply by following a method due to Murphy [5] who solved the problem in electrostatics of determining the potential of two charged spheres, namely by determining not the coordinates a_{in} themselves, but their products. Let us introduce new coefficients A^i_n and B^i_n defined by

$$2\alpha^i_{2n} = -u_i (R_i / A^i_n)^3, \quad 2\alpha^i_{2n-1} = -u_k (R_k / B^i_n)^3 \tag{1.7}$$

Then by (1.6) the dipole coordinates are

$$a_{i2n} = R_i B^k_n / A^i_n, \quad a_{i2n-1} = R_i A^k_{n-1} / B^i_n \tag{1.8}$$

Coefficients A^i_n and B^i_n are obtained from (1.5)-(1.8) in the form

$$(\tau - \tau^{-1}) A^i_n = \tau^n (\tau + R_i / R_k) - \tau^{-n} (\tau^{-1} + R_i / R_k),$$

$$(\tau - \tau^{-1}) B^i_n = (\tau^n - \tau^{-n}) a / R_i \tag{1.9}$$

where τ is a root of the equation

$$a^2 \tau = (\tau R_1 + R_2)(\tau R_2 + R_1) \tag{1.10}$$

Indeed, inserting (1.8) into (1.5) we obtain the following recurrent relations

$$R_i B^i_n + R_k B^i_{n-1} = a A^k_{n-1}, \quad R_i A^i_n + R_k A^i_{n-1} = a B^k_n \tag{1.11}$$

with initial conditions

$$A^i_0 = 1, \ A^i_1 = (a^2 - R_k^2) / R_1 R_2; \quad B^i_0 = 0, \ B^i_1 = a / R_i \tag{1.12}$$

Relations (1.11) are solved for A_n^i and B_n^i. In particular, for A_n^i we have

$$A_n^i - A_{n-1}^i (a^2 - R_1^2 - R_2^2) / R_1R_2 + A_{n-2}^i = 0 \tag{1.13}$$

and an analogous equation can be obtained for B_n^i. In the case of two spheres, the general solution of the recurrent relation (1.13) with arbitrary conditions is given by $c_1\tau^n + c_2\tau^{-n}$ where τ is defined by (1.10). The constants c_1 and c_2 are obtained from the initial conditions (1.12) and the final result has the form of (1.9).

The series (1.1) whose functions are defined by (1.4) is convergent everywhere except at the point $\theta_i = 0$, $r_i = R_i$ of contact of the spheres. In the latter case the point a_{in} of accumulation of the dipole coordinates passes into the point of contact of the spheres. This is precisely the reason why the method of expanding the series in terms of spherical functions used in [2] fails, when the distance between the spheres is small compared with the radius. When the point of contact is deleted, the potential series converges approximately as $1 / n^3$. When the spheres do not touch, then from (1.5) and (1.6) it follows that the series converges approximately like a geometric series whose exponent decreases rapidly with the increasing distance between the spheres.

2. Tangential velocity on the surface of the spheres in contact. When (1.5) and (1.6) are taken into account, the potential formulas (1.1) and (1.4) on the surface of the spheres become much simpler and are

$$\Phi|_{R_i} = \frac{\alpha_0^i \cos\theta_i}{R_i^2} + \sum_{n=1}^{\infty} \alpha_n^i \left(\frac{R_i^2}{a_{in}} - a_{in}\right) (R_i^2 - 2R_i a_{in} \cos\theta_i + a_{in}^2)^{-3/2} \tag{2.1}$$

When the spheres are in contact, (1.10) gives $\tau = 1$ and (1.9) yields

$$B_n^i = na / R_i, \qquad A_n^i = 1 + na / R_k \tag{2.2}$$

Insertion of (2.2) into (1.8) gives the dipole coordinates

$$a_{i2n} = R_i / (1 + R_k / an), \qquad a_{i2n-1} = R_i (1 - R_k / an) \quad (n = 1, 2, \ldots) \tag{2.3}$$

Since the problem is linear, it is sufficient to consider separately the collision of the spheres and their motion in the same direction. Assuming therefore that $u_1 = \pm u_2$ and introducing the variable $\xi = \operatorname{tg} {}^1/_2\theta$, we obtain (from (2.1)–(2.3) and (1.7), (1.8)) the following expression for the tangential velocity v_θ at the surface of a sphere

$$v_\theta = \frac{1}{2} u_i \sin\theta_i + \frac{3}{2} u_i \sin\theta_i (1 + \xi^2)^{5/2} \times$$

$$\times \sum_{n=1}^{\infty} \left\{ \frac{n\gamma + 1}{[1 + (n\gamma + 1)^2\xi^2]^{5/2}} \pm \frac{n\gamma - 1}{[1 + (n\gamma - 1)^2\xi^2]^{5/2}} \right\}, \qquad \gamma = \frac{2(R_1 + R_2)}{R_k} \tag{2.4}$$

Here the plus sign corresponds to the spheres moving towards each other with equal velocities and the minus sign corresponds to their motion in the same direction.

In the first case, the asymptotic behavior of the series appearing in (2.4) as $\xi \to 0$ $(\theta \to 0)$ is described by $v_\theta = 2u_i R_k / (R_1 + R_2)\theta_i + O(\mathrm{const})$ which is easily obtained using the Euler-Maclaurin formula.

Thus when the spheres approach each other up to the point of contact, a plane source is formed at this point and ejects the fluid into the tangential plane.

When the velocities of the spheres in contact have the same direction, i.e. when $u_1 = -u_2$, we find it convenient to use a coordinate system moving with the spheres. Relation (2.4) will then become

$$v_\theta = -{}^3/_2\,(1+\xi^2)^{1/2}\; f(\xi)\; u_i \sin\theta_i \qquad (\xi = \mathrm{tg}\ {}^1/_2\theta) \tag{2.5}$$

$$f(\xi) = \sum_{n=-\infty}^{\infty} g(\xi, n - 1/\gamma), \qquad g(\xi, x) = \gamma x\,[1 + (x\gamma\xi)^2]^{-5/2} \tag{2.6}$$

We shall show later that for $\xi \to 0$,

$$f(\xi) = -\frac{2\pi}{3}(4\pi + {}^3/_4\gamma\xi)\left(\sin\frac{2\pi}{\gamma}\right)\exp\left(-\frac{2\pi}{\gamma\xi}\right)/\,\gamma^{5/2}\xi^{7/2} \tag{2.7}$$

From the formulas (2. 5) and (2. 7) we see that the tangential velocity at the surface of a sphere decreases exponentially with decreasing distance to the point of contact. When the radii of the spheres are equal, the velocity near the point of contact varies as $\exp(-\pi/\theta)/\theta^{5/2}$ and the fluid stagnates near the point of contact.

The asymptotic formula (2. 7) is obtained from (2. 6) by means of the following Poisson's summation formula [6]:

$$f(\xi) = \sum_{n=-\infty}^{\infty} g(\xi, n - 1/\gamma) = \sum_{l=-\infty}^{\infty} e^{-2\pi i l/\gamma} \int_{-\infty}^{\infty} e^{-2\pi i l x} g(\xi, x)\,dx \tag{2.8}$$

Let us denote the integrals in (2. 8) by I_l. Since $g(\xi, x) = -g(\xi, -x)$, we have $I_l = -I_{-l}$. We make the substitution $x\xi\gamma = t$ in the integrals, and we have

$$I_l = \frac{1}{\gamma\xi^2}\int_{-\infty}^{\infty} t\,(1+t^2)^{-5/2} e^{i\sigma t}\,dt \qquad \left(\sigma = -\frac{2\pi l}{\xi\gamma}\right) \tag{2.9}$$

Before changing to the new contour of integration, we must integrate (2. 9) by parts, and modify the resulting expression for the integrand at the point $t = i$ with $\sigma > 0$ in such a manner, that the integral along the imaginary axis to the point $t = i$ will be convergent. The expression for I_l then becomes

$$I_l = \frac{i\sigma}{3\gamma\xi^2}\int_{-\infty}^{\infty}\left[\frac{1+it}{(1+t^2)^{3/2}} + \frac{\sigma}{(1+t^2)^{1/2}}\right] e^{i\sigma t}\,dt \tag{2.10}$$

To transform (2.10) we choose the following contour: $-R$ to $+R$ along $\mathrm{Im}\,t = 0$, a circular arc $Re^{i\theta}$, $\theta \in [0, {}^1/_2\pi]$, a segment from $iR + \varepsilon$ to $i + \varepsilon$, a circular arc $\varepsilon e^{i\theta} + i$, $\theta \in [-\pi, 0]$, a segment from $i - \varepsilon$ to $iR - \varepsilon$ and a circular arc $Re^{i\theta}$, $\theta \in [{}^1/_2\pi, \pi]$ (R and ε are real numbers). The function $(1 + t^2)^{1/2}$ assumes the value of $-i\,(y^2 - 1)^{1/2}$ on the segment $iy - \varepsilon$ and the value of $i\,(y^2 - 1)^{1/2}$ on the segment $iy + \varepsilon$. When $R \to \infty$ and $\varepsilon \to 0$, we have by the Cauchy's theorem

$$I_l = \frac{2i\sigma}{3\gamma\xi^2}\int_1^{\infty}\frac{1 + \sigma(y+1)}{(1+y)(y^2-1)^{1/2}}\,e^{-\sigma y}\,dy$$

Performing the change of variable $y = 1 + u^2$ and expanding the resulting integrand function into a series, we obtain the first terms of the asymptotic formula for I_l as $\sigma \to \infty$ ($\xi \to 0$)

$$I_l = {}^1/_3 i\pi\,\sqrt{-l}\,({}^3/_4\xi\gamma - 4\pi l)\exp\left(\frac{2\pi l}{\xi\gamma}\right)\gamma^{-5/2}\xi^{-7/2} \tag{2.11}$$

where we have assumed that $\sigma = -2\pi l/\xi\gamma$. The formula (2. 11) is valid only for $l < 0$. Utilizing the fact that I_l is an odd function of l and making a single assumption that $l = \pm 1$, we obtain from (2. 11) and (2. 8) the formula (2. 7).

3. Kinetic energy of the fluid. As we know [3, 4], the kinetic energy T of a perfect incompressible fluid can be expressed in terms of the potential of its boundaries

$$\frac{2}{\rho} T = \int_S \Phi \frac{\partial \Phi}{\partial n} ds \tag{3.1}$$

Motion of two spheres can always be represented as the sum of motions in three, mutually perpendicular directions, one of which coincides with the line-of-centers. Kinetic energy of the fluid at an arbitrary motion of the spheres is equal to the sum of its components appearing in the above motion [1, 2]. This additive property can be proved either by symmetry considerations [1, 2], or proceeding from the simplest potential properties and Green's identities. The additive property of kinetic energy enables us to compute it for just two cases: motion of the spheres along the line-of-centers, and their motion in the direction perpendicular to this line when the velocities are coplanar.

When the only motion is that along the line-of-centers, insertion of (1.2) and (2.1) into (3.1) following the computation of the integral

$$\int_0^\pi \frac{(R_i^2 - a_{in}^2)\cos\theta\sin\theta}{a_{in}(R_i^2 - 2R_i a_{in}\cos\theta + a_{in}^2)^{3/2}} d\theta = \frac{2}{R_i^2}$$

yields the following expression for the kinetic energy of the fluid

$$\frac{1}{2\pi\rho} T = -\sum_{i=1}^{2} u_i \left(\frac{1}{3}\alpha_0^i + \sum_{n=1}^{\infty} \alpha_n^i\right) \tag{3.2}$$

This problem was solved by Hicks in a somewhat different manner [1, 2]. Here α_n^i are given by (1.7) and (1.9) as functions of τ, and (1.10) connects a with τ.

Kinetic energy is a quadratic form of the velocities

$$\frac{1}{\pi\rho} T = A_1 u_1^2 + 2Bu_1u_2 + A_2u_2^2 \tag{3.3}$$

The coefficients A_i and B can be written, in accordance with (1.7) and (3.3), as

$$\frac{A_i}{R_i^3} = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{1}{(A_n^i)^3}, \qquad \frac{B}{R_k^3} = \sum_{n=1}^{\infty} \frac{1}{(B_n^i)^3} \tag{3.4}$$

where A_n^i and B_n^i are given by (1.9).

When the spheres are in contact, the coefficients A_i and B become particularly simple, provided that (2.2) is taken into account

$$\frac{A_i}{R_i^3} = \frac{1}{3} + \sum_{n=1}^{\infty}\left(1 + n\frac{R_1 + R_2}{R_k}\right)^{-3}, \qquad B = \zeta(3)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-3}$$

where $\zeta(x)$ denotes the Riemann's zeta function. In particular, when both spheres are of equal radius, we easily obtain

$$A = R^3\left({}^7/_8\zeta(3) - {}^2/_3\right) \approx 0.385\, R^3, \qquad B = 0.125\zeta(3)R^3 \approx 0.150\, R^3$$

coinciding with the analogous result given in [1].

4. Forces of hydrodynamic interaction between two spheres at small distances. Since the motion of spheres in a perfect incompressible fluid can be described in terms of Lagrange's equations [3, 4], it follows that $\partial T / \partial a$ denotes the force of hydrodynamic interaction between two spheres. Hicks established [1] that the series defining the coefficients of the quadratic form $\partial T / \partial a$ become divergent when the spheres touch each other. Two leading terms of the asymptotic expression for the sums of series can be obtained in the following manner. Denoting by $f(n, \tau)$ the general term of one of the series for dA_i / da or dB / da obtained by differentiation of (3.4) term by term, we note that the functions

$$\sum_{n=E[1/(\tau-1)]}^{\infty} f(n, \tau), \qquad \sum_{n=1}^{E[1/(\tau-1)]} [f(n, \tau) - f(n, 1)]$$

where $E(1 / (\tau - 1)$ denotes the integral part of $(\tau - 1)^{-1}$, are bounded when $\tau \to 1$. Moreover, since $f(n, \tau)$ tends to zero uniformly in τ as $n \to \infty$, the difference $f(n, \tau) - f(n, 1)$ tends to zero uniformly in n as $\tau \to 1$ despite the fact that $f(n, \tau)$ is not uniformly continuous in τ. By virtue of the above remarks, the last two series in the identity

$$\sum_{n=1}^{\infty} f(n, \tau) = \sum_{n=1}^{E[1/(\tau-1)]} f(n, 1) + \sum_{n=1}^{E[1/(\tau-1)]} [f(n, \tau) - f(n, 1)] + \sum_{n=E[1/(\tau-1)]}^{\infty} f(n, \tau)$$

can be replaced by integrals. Consequently the formula

$$\sum_{n=1}^{\infty} f(n, \tau) = \sum_{n=1}^{E[1/(\tau-1)]} f(n, 1) - \int_1^{1/(\tau-1)} f(x, 1)\, dx + \int_1^{\infty} f(x, \tau)\, dx + O(\tau - 1) \tag{4.1}$$

holds.

Applying the latter formula to the series (3.4) defining the coefficients of $\partial T / \partial a$ we find that the series in the right side of (4.1) decomposes easily into a divergent series plus a constant. The integrals appearing in (4.1) are computed and only the values of the two highest order terms with respect to $(\tau - 1)$ are retained in the resulting expressions. After several tedious manipulations we finally obtain

$$p = R_1 R_2 / (R_1 + R_2), \qquad d = {}^2/_3 - {}^1/_2 \ln 2 - c, \quad \delta = a - R_1 - R_2$$

$$\frac{1}{p^2} \frac{dA_i}{da} = d + \frac{1}{2} \ln \frac{\delta}{p} - \sum_{n=1}^{\infty} \left[\frac{n(n+1)(n-1+3p/R_i)}{(n+p/R_i)^4} - \frac{1}{n} \right] \tag{4.2}$$

$$\frac{1}{p^2} \frac{dB}{da} = d + \frac{1}{2} \ln \frac{\delta}{p} + (1 - 3p^2/R_1 R_2)\, \zeta(3)$$

where c is the Euler's constant. When two spheres of equal radius move towards each other with equal velocities, at small distances we have, by (4.2), (3.3) and (3.4)

$$\partial T / \partial a \approx [{}^1/_2 \ln (a / R - 2) - 0.0948]\, \pi \rho u^2 R^2 \tag{4.3}$$

Formulas (4.2) show clearly that the difference $dA_i / da - dB / da$ remains finite when $a \to R_1 + R_2$, i.e. when the spheres are about to touch each other. It can be shown that this difference is always positive, i.e. spheres moving in the same direction draw apart irrespective of the values of their radii. From (3.3), (3.4) and (4.2) it follows that two spheres of equal radii in contact and moving in the same direction will repel each other with a force equal to

$$\partial T / \partial a = ({}^3/_4 \zeta(3) - \ln 2)\, \pi \rho u^2 R^2 \approx 0.2084 \pi \rho u^2 R^2 \qquad (4.4)$$

5. Velocity potential in the case of spheres moving in the direction perpendicular to the line-of-centers. When the spheres move in the direction perpendicular to the line-of-centers ($u_1 = u_2 = 0$), the zeroth approximation for the potential, satisfying (1.2), is given in the ith coordinate system in the form

$$\Phi_0^i = -\frac{R_i^3}{2r_i^2} v_i \sin\theta_i \sin\varphi_i, \quad \Phi_0^k = -R_k^3 v_k \frac{r_i \sin\theta_i \sin\varphi_i}{2(r_i^2 - 2ar_i\cos\theta_i + a^2)^{3/2}} \qquad (5.1)$$

As we know [1, 2], the potential can be represented by a set of dipoles situated within the spheres along the line-of-centers and orthogonal to it. Our problem is to determine this set. To solve this problem it is expedient to introduce the coordinates of the dipoles as functions of the dimensionless variables x_n. The dipole coordinate $b_{in} = b_{in}(a, x_1, \ldots, x_n)$ can be found analogously to that of (1.5)

$$b_{in} = R_i^2 x_n (a - b_{kn-1})^{-1}, \quad b_{i0} = 0, \quad x_n \in [0,1] \quad (n = 1, 2, \ldots) \qquad (5.2)$$

Clearly, we always have $b_{in} \leqslant a_{in}$, whereas $b_{in} = a_{in}$ only when $x_1 = 1, \ldots, x_n = 1$. We can write

$$Q_n^i = r_i \sin\theta_i \sin\varphi_i (r_i^2 - 2r_i b_{in}\cos\theta_i + b_{in}^2)^{-3/2} \qquad (5.3)$$

to describe the dipole in the ith sphere whose coordinate is b_{in}.

It can be shown that for fixed n, Eq. (1.3) is satisfied by the functions

$$\Phi_{n-1}^k = Q_{n-1}^k, \quad \Phi_n^i = \left(\frac{b_{in}}{R_i}\right)^3 \left(Q_n^i - \int_0^1 Q_n^i x_n\, dx_n\right) \qquad (5.4)$$

where $x_n = 1$ for the functions appearing outside the integral sign. We easily see that any Φ_n^i appearing in (1.3) can be constructed by applying the formulas (5.4) n times to the functions given in their zeroth approximation by (5.1). Function Φ_n^i on the other hand, requires the introduction of the coefficients $\beta_n^i = \beta_n^i(a, x_1, \ldots, x_{n-1})$ which represent a generalization of the coefficients α_n^i appearing in the course of solution of the problem of motion of the spheres along the line-of-centers

$$2\beta_{2n}^i = v_i R_i^{3-3n} R_k^{-3n} (a_{k1} b_{i2} b_{k3} \ldots b_{i2n})^3 \qquad (5.5)$$
$$2\beta_{2n-1}^i = v_k R_i^{-3n} R_k^{6-3n} (a_{i1} b_{k2} b_{i3} \ldots b_{i2n-1})^3$$

(here the argument $x_m = 1$ in every b_{im} $(m = 1, 2, \ldots)$).

We note that $|\beta_n^i| \leqslant |\alpha_n^i|$ and, that the equality sign appears only when $x_1 = 1, \ldots, x_{n-1} = 1$. The new coefficients must be supplemented by the operator L_n defined by

$$L_n f(x_n) = f(1) - \int_0^1 f(x_n)\, x_n\, dx_n \quad (n=1,2,\ldots) \tag{5.6}$$

Using now (5.1) and (5.4)–(5.6), we can write Φ_n^i in a compact, analytic form

$$\Phi_0^i = -{}^1/_2 v_i R_i{}^3 Q_0^i, \qquad \Phi_n^i = -L_1 \ldots L_{n-1}\beta_n^i L_n Q_n^i \tag{5.7}$$

and this completes the solution of the problem of obtaining the velocity potential.

To compute the kinetic energy of the fluid, it is sufficient to know the potential Φ on the spheres. The latter assumes a simpler form if we take into account the fact that

$$Q_{n-1}^k = (b_{in} / R_i)^3 Q_n^i \quad \text{when } r_i = R_i$$

and consequently, $\Phi_{n-1}^k = -L_1 \ldots L_{n-1}\beta_n^i\ Q_n^i$. Then, by (1.3) and (5.7), the potential on a sphere is

$$\Phi\,|_{R_i} = -\frac{1}{2} v_i R_i{}^3 Q_0^i\,|_{R_i} - \sum_{n=1}^{\infty} L_1 \ldots L_{n-1}\beta_n^i \left(2Q_n^i - \int_0^1 Q_n^i x_n\, dx_n\right)\Big|_{R_i} \tag{5.8}$$

6. Kinetic energy of the fluid when the spheres move in the direction perpendicular to the line-of-centers. When the spheres move perpendicularly to the line-of-centers and their velocities are coplanar ($u_1 = u_2 = 0$), the kinetic energy is obtained from (1.2), (3.1), (5.3) and (5.8). An assumption that

$$\int_{S_i} \left(2Q_n^i - \int_0^1 Q_n^i x_n\, dx_n\right) \sin\theta_i \sin\varphi_i\, ds = 2\pi$$

is sufficient to obtain the kinetic energy in the form

$$\frac{1}{\pi\rho} T = \sum_{i=1}^{2} \left(\frac{1}{3} v_i^2 R_i{}^3 + \sum_{n=1}^{\infty} L_1 \ldots L_{n-1}\beta_n^i v_i\right) \tag{6.1}$$

where the operator L_n is given by (5.6), the coefficient β_n^i is known from (5.5) and L_0 is equal to the unit operator.

It can be proved that the series in the right side of (6.1) converges faster than the corresponding series in the right side of (3.2), the latter converging approximately as $1 / n^3$. First, we show that the continued fraction b_{in} defined by (5.2) can be expanded into a $(n - 1)$-dimensional convergent power series in $x_1, \ldots, x_{n-1}$ with nonnegative coefficients. This can easily be done by expanding consecutively the continued fractions $b_{i1}, b_{i2}, \ldots$ into a series using the formula (5.2). The fact that all b_{in} can be expanded into converging power series with nonnegative coefficients in an $(n - 1)$-dimensional hypercube $x_1 \in [0, 1], \ldots, x_{n-1} \in [0, 1]$ implies, that $|\beta_n^i|$ defined by (5.5) can also be expanded into a convergent power series with nonnegative coefficients, in the same hypercube

$$|\beta_n^i| = \sum_{m_1,\ldots,m_{n-1}} C^i_{m_1,\ldots,m_{n-1}} x_1^{m_1} \ldots x_{n-1}^{m_{n-1}}, \qquad C^i_{m_1,\ldots,m_{n-1}} \geqslant 0 \qquad (6.2)$$

Simple computations based on (5.6) and (5.2) yield

$$0 < |L_1 \ldots L_{n-1}\beta_n^i| = \sum_{m_1,\ldots,m_{n-1}} C^i_{m_1,\ldots,m_{n-1}} \frac{(m_1+1)\ldots(m_{n-1}+1)}{(m_1+2)\ldots(m_{n-1}+2)} <$$
$$< \sum_{m_1,\ldots,m_{n-1}} C^i_{m_1,\ldots,m_{n-1}} = |\alpha_n^i|$$

where α_n^i is defined by (1.6) for $u_i = v_i$. The inequalities obtained show clearly that the kinetic energy expressed by the series (6.1) is majorized by the kinetic energy expressed by the series (3.2) for any distance between the spheres, provided that $u_i = v_i$. Consequently, the series for kinetic energy converges faster in the case of motion perpendicular to the line-of-centers, than in the case of a motion along this line.

Formulas (5.5), (5.6) and (6.1) yield the coefficients A_i' and B' appearing in the expression for kinetic energy $T = A_1'v_1^2 + 2B'v_1v_2 + A_2'v_2^2$. For the spheres of equal radii, these formulas yield

$$A' = 0.347\pi\rho R^3, \qquad B' = 0.067\pi\rho R^3$$

for the case of a contact.

When two equal spheres in contact move with equal velocities in the direction perpendicular to the line-of-centers, the kinetic energy of the fluid is found to be equal to $T = 0.828\pi\rho u^2 R^3$.

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